Qu 1

Thursday, March 24, 2016 3:56 PM

(a) $\operatorname{diam}(\mathring{A}) < \operatorname{diam}(A)$ in general.

Example. $A = \mathbb{Q} \cap [0,1]$ in $X = [0,1]$

Use ε-ball argument to show that

$\forall \varepsilon > 0 \quad \operatorname{diam}(\bar{A}) \leq \operatorname{diam}(A) + 2\varepsilon$

Since $\operatorname{diam}(A) \leq \operatorname{diam}(\bar{A})$ is obvious,

$$\operatorname{diam}(A) = \operatorname{diam}(\bar{A})$$

(b) As $d(x,A)$ is a supremum,

$d(x,A) = 0 \Rightarrow \forall \varepsilon > 0 \ \exists\, a \in A$,

$d(x,a) < \varepsilon$

That is exactly $x \in \bar{A}$.

(c) Try to apply Δ-inequality to the supremum $d(x,A)$.

(d) The set is open.

Note that $f: X \longrightarrow \mathbb{R}$ is continuous,

$$f(x) = d(x,A) - d(x,B)$$

The set is $f^{-1}(0,\infty)$.

Qu 2

Friday, April 15, 2016 5:28 PM

(a) Note that $\mathcal{T}_{cf} \subset \mathcal{T}_{std} \subset \mathcal{T}_{ll}$

Use it to logically argue that

$x_n \to x$ in $\mathbb{R}_{ll}$

$\Rightarrow x_n \to x$ in $\mathbb{R}_{std}$

$\Rightarrow x_n \to x$ in $\mathbb{R}_{cf}$

(b) Show that in $\mathbb{R}_{cf}$, every infinite sequence converges to any point in $\mathbb{R}_{cf}$. Obviously, pick one that does not converge in $\mathbb{R}_{std}$, which also diverges in $\mathbb{R}_{ll}$.

Also, a sequence $x_n < a$ and $x_n \to a$ in $\mathbb{R}_{std}$ does not converge in $\mathbb{R}_{ll}$

(c) Such example does not exist

First, any $f: X \to Y$ with 1^{st} countable X satisfying "$x_n \to x$ implies $f(x_n) \to f(x)$" must be continuous

So, no required example with $X = \mathbb{R}_{std}$ nor $X = \mathbb{R}_{ll}$.

Qu2 continued

Tuesday, April 19, 2016 6:03 PM

Second, we consider $f: \mathbb{R}_{cf} \longrightarrow Y$ where
$Y = \mathbb{R}_{cf}$ or $\mathbb{R}_{std}$ or $\mathbb{R}_{ll}$

Claim: Let f be discontinuous. Then $\exists$
$x_n \to r$ in $\mathbb{R}_{cf}$ but $f(x_n) \not\to f(r)$.

By this claim, no required example for $X = \mathbb{R}_{cf}$

Since f is not continuous, $\exists\, V \in \mathcal{T}_{cf}$ or $\mathcal{T}_{std}$ or $\mathcal{T}_{ll}$
$f^{-1}(V) \notin \mathcal{T}_{cf}$, i.e., $\mathbb{R} \setminus f^{-1}(V)$ is infinite

Pick a distinct sequence $p_n \in \mathbb{R} \setminus f^{-1}(V)$,
so $f(p_n) \notin V$.

On the other hand, $f^{-1}(V) \in \mathcal{T}_{cf}$, $\therefore f^{-1}(V) \neq \emptyset$
$\exists\, r \in f^{-1}(V)$, i.e. $f(r) \in V$

Define a sequence x_n in $\mathbb{R}_{cf}$ by
$x_{2n} = p_n$ and $x_{2n+1} = r$

$(x_n)_{n=1}^{\infty}$ is an infinite sequence, $x_n \to r$ in $\mathbb{R}_{cf}$

However, $f(x_{2n}) \notin V$ while $f(x_{2n+1}) = f(r) \in V$
where $V \in \mathcal{T}_{cf}$ or $\mathcal{T}_{std}$ or $\mathcal{T}_{ll}$

Thus, $f(x_n) \not\to f(r)$ in $\mathbb{R}_{cf}$, $\mathbb{R}_{std}$, $\mathbb{R}_{ll}$.

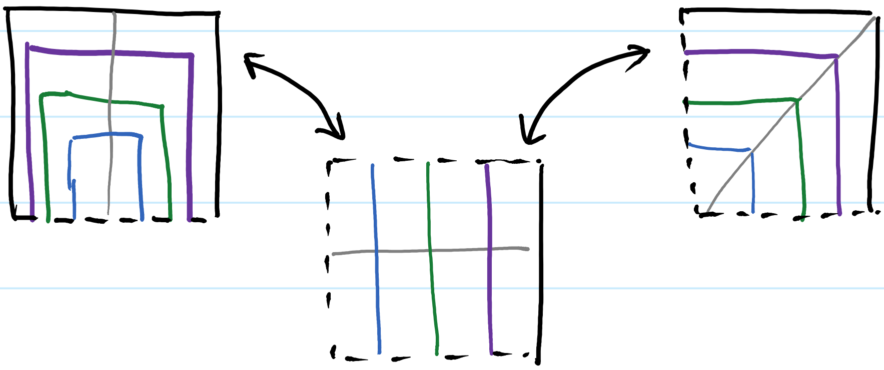
(d) Intersecting $(a,b) \times [c,d)$ with the diagonal $\{(x,x) : x \in \mathbb{R}\}$, we get a base for lower-limit topology.

Qu3

Tuesday, April 19, 2016 6:37 PM

(a) There may be several ways to write the homeomorphism, one possible idea is as below.

$(0,1]\times[0,1]$ $(0,1]\times(0,1]$



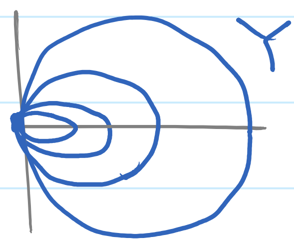
$(0,1]\times(0,1)$

(b) Basic nbhd of $\bar{0}\in\mathbb{R}^N$ is of the form

$$U=\underbrace{\mathbb{R}\times\cdots\times(-\varepsilon_1,\varepsilon_1)\times\mathbb{R}\times\cdots\times\mathbb{R}\times(-\varepsilon_p,\varepsilon_p)}_{N}\times\mathbb{R}\times\cdots\times\mathbb{R}\times\cdots$$

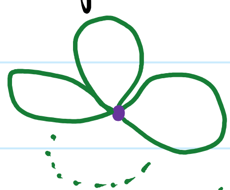
Show that for this N, if $n>N$ then $\bar{a}_n\in U$ by definition of $\bar{a}_n$

(c) $\mathbb{R}/\sim\ \neq Y$



All $n\in\mathbb{Z}$ are identified as a single point.

$\mathbb{R}/\sim\ =$



Qu3 continued

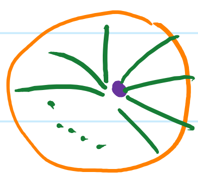
Wednesday, April 20, 2016 4:11 PM

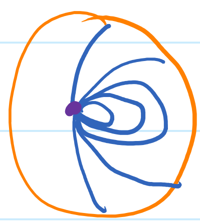
The crucial problem occurs at

$[n] \in \mathbb{R}/\sim$, $n \in \mathbb{Z}$ and $(0,0) \in Y \subset \mathbb{R}^2$

Arbitrary nbhds at the two points

at $[n]$ in $\mathbb{R}/\sim$, at $(0,0) \in Y$





(d) To prove $\mathcal{T}_q$ satisfies QT4, just consider

$$(f \circ q)^{-1}(W) = q^{-1}(f^{-1}(W)) \text{ for } W \in \mathcal{T}_Z$$

To prove $\mathcal{T}_q$ is the unique one, consider

$$f = \mathrm{id} : (X/\sim, \mathcal{T}) \longrightarrow (X/\sim, \mathcal{T}_q)$$

and conclude that $\mathcal{T} = \mathcal{T}_q$.

Qu4

Wednesday, April 20, 2016 4:52 PM

(a) $f : (B[a,b], d_\infty) \longrightarrow (\mathbb{R}, \text{std})$

is uniformly continuous.

Idea: apply Δ-inequality to

$|f(x_1) - f(x_2)|$ for $x_1, x_2 \in B[a,b]$

$= |d_\infty(x_1, 0) - d_\infty(x_2, 0)|$

(b) Use completeness of $\mathbb{R}$ to get pointwise convergence first.

(c) A is open $\Leftrightarrow$ $X \setminus A$ is closed

$\Leftrightarrow$ $\overline{X \setminus A} = X \setminus A$

$(\overline{X \setminus A})^\circ = (X \setminus A)^\circ \subset X \setminus \bar{A}$

(d) It is of 2^{nd} category.

The only nowhere dense set is ϕ.

Singleton is not because $(\overline{\{n\}})^\circ = \{n\}^\circ = \{n\}$.